

Monotonicity of a Class of Integral Functionals

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Abstract

In this note we prove a condition of monotonicity for the integral functional $F(g) = \int_a^b h(x) d[-g(x)]$ with respect to g , a function of bounded variation.

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1 Introduction

In the article [1] (“*Nontrivial Equilibria of a Quasilinear Population Model*”, in progress), I study a functional $R(u)$ ($u \in L^1(0, \infty)$), said *generalized net reproduction rate*, to prove existence of non-zero equilibria in a general structured population model.

The monotonicity of $R(u)$ is used in a Corollary to prove the non-existence of a non-zero stationary population if $R(0) < 1$ (a sufficient condition of existence being $R(0) > 1$).

The original proposition about monotonicity, not so immediate, will be reduced to the integration by parts of an improper Stieltjes integral:

$$\int_a^\infty h(x) d[-g(x)] = h(a)g(a) - \lim_{b \rightarrow \infty} h(b)g(b) + \int_a^\infty g(x) dh(x)$$

2 Monotonicity Propositions

Assume $0 < a < b \leq \infty$.

From now on we denote via $G(b)$ the value of $G(b)$ if $b < \infty$ and $\lim_{x \rightarrow \infty} G(x)$ if $b = \infty$. I will denote respectively in the cases $[a, b]$ and $[a, \infty)$.

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Proposition 1 *Let H, G be two given functions on I .*

Let H be increasing (non-decreasing), bounded, non-negative. Let G be continuous and of bounded variation.

Define

$$\mathcal{F}(G) := \int_a^b H(x) d[-G(x)]. \quad (1)$$

If $G(b) = 0$, then \mathcal{F} is increasing (non-decreasing) with respect to G , i.e. let be $A := \{\phi | \phi \in C([a, b]) \cap BV[a, b], \phi(b) = 0\}$: if $G_1, G_2 \in A$ and $G_1 < G_2$, then $\mathcal{F}(G_1) < \mathcal{F}(G_2)$ (respectively $\mathcal{F}(G_1) \leq \mathcal{F}(G_2)$).

Proof. a) Consider first the case $b < \infty$. $\mathcal{F}(G)$ is well-defined; integrating by parts we have:

$$\mathcal{F}(G) = -H(b)G(b) + H(a)G(a) + \int_a^b G(x) dH(x) = H(a)G(a) + \int_a^b G(x) dH(x). \quad (2)$$

The conclusion is immediate.

b) Consider the case $b = \infty$. For H bounded and $G(x)$ converging for $x \rightarrow \infty$ we obtain immediately the existence of the improper integral and extend the formula of case a).

If $H(x)$ is not strictly increasing but only non-decreasing, the functional \mathcal{F} is only non-decreasing with respect to G .

Corollary 2 *Let H, G given functions on I .*

Let H be decreasing (non-increasing), bounded, non-negative. Let G be continuous and of bounded variation.

Define $\mathcal{F}_0(G) := \int_a^b H(x) dG(x)$.

If $G(b) = 0$, then \mathcal{F}_0 is increasing (non-decreasing) with respect to G .

Example 1. Consider the functional

$$\mathcal{I}(f) = \int_0^\infty dx h(x) f(x) e^{-\int_0^x dy f(y)} \quad (3)$$

where h is positive, increasing and bounded. If $f \in L^1_{loc}(0, \infty)$, $f \geq 0$ and $\int_0^\infty dy f(y) = \infty$ ($f \notin L^1(0, \infty)$), then \mathcal{I} is decreasing with respect to f .

This is a particular case of Prop. 1, where $g(x) = e^{-\int_0^x dy f(y)}$ and

$$\mathcal{I}(f) = \int_0^\infty dx h(x) d[-e^{-\int_0^x dy f(y)}].$$

Corollary 3 *Consider $u \in L^1(0, \infty)$ and the functional*

$$R(u) = \int_0^\infty h(x, u(\cdot)) f(x, u(\cdot)) e^{-\int_0^x dy f(y, u(\cdot))} \quad (4)$$

where h and f are defined from $(0, \infty) \times L^1_+(0, \infty)$ in $[0, \infty)$, h is positive and bounded, $x \mapsto f \in L^1_{loc}(0, \infty)$ and $\int_0^\infty dy f(y) = 0$, and

- let $x \mapsto h(x, u)$ be non-decreasing (increasing) for fixed u
- $u \mapsto h(x, u)$ decreasing (o non-increasing) for fixed x
- $u \mapsto f(x, u)$ non-decreasing (o increasing) for fixed x

Then $R(u)$ is decreasing with respect to u .

Proof. Take $u_1, u_2 \in L_+^1(0, \infty)$ with $u_1 < u_2$. For Proposition 1, the integral

$$\int_0^\infty h(x, u_1) f(x, u) e^{-\int_0^x dy f(y, u)}$$

is decreasing with respect to f , that is non-decreasing in u : therefore this integral is non-increasing in u and we have

$$\int_0^\infty h(x, u_1) f(x, u_1) e^{-\int_0^x dy f(y, u_1)} \geq \int_0^\infty h(x, u_1) f(x, u_2) e^{-\int_0^x dy f(y, u_2)}. \quad (5)$$

As f is decreasing with respect to u , we have

$$\int_0^\infty h(x, u_1) f(x, u_2) e^{-\int_0^x dy f(y, u_2)} > \int_0^\infty h(x, u_2) f(x, u_2) e^{-\int_0^x dy f(y, u_2)}, \quad (6)$$

so that $R(u_1) > R(u_2)$.

(The case of the alternative conditions, given by the parenthesis, is analogous).

Example 2. Corollary 3 is applied to a model of population dynamics: let $u = u(t, x) \geq 0$ be a population density with respect to age or size $x \geq 0$. Existence of stationary solutions (i. e. equilibria) $u = u(x)$ is related to a functional $R(u)$, the net reproduction rate. In a generalized model (see [1]) where g and μ depend on u in an infinite-dimensional kind, $R(u)$ is represented by

$$R(u) = \int_0^\infty dx \beta(x, u(\cdot)) \frac{e^{-\int_0^x dy \frac{\mu(y, u(\cdot))}{g(y, u(\cdot))}}}{g(x, u(\cdot))} \quad (7)$$

where β represents fertility, μ mortality and g is a coefficient of growth (the detailed model is given and discussed in [1]).

The condition of existence of a nonzero steady solution (with suitable regularity conditions) is requiring that $R(u) = 1$; see [2, 3] and [1]. See also [4, 5, 8].

If $R(0) < 1$ and monotonicity conditions hold, the zero solution is the unique equilibrium.

I prove in [1] that $R(0) > 1$ is a sufficient condition for existence of nontrivial stationary solutions. If monotonicity conditions do not hold, then $R(0) > 1$ is sufficient but it is not necessary and it is simple to give a counterexample.

3 More about the Application

The model is a generalized version of the classic Lotka-MacKendrick population model: consider a population density $u = u(t, x)$, where $t \in [0, T]$ represents *time*, $x \in (0, \infty)$ is *age* or *size* and the total population $P(t)$ is

$$P(t) = \int_0^\infty u(t, x) dx.$$

Consider the following functions: growth/diffusion $g = g(x, u)$, mortality $\mu = \mu(t, u)$, fertility $\beta = \beta(x, u)$, depending on x and infinite-dimensionally depending on the population density $u(t, \cdot)$. The model is

$$u_t(t, x) + (g(x, u(t, \cdot)) u(t, x))_x + \mu(x, u(t, \cdot)) u(t, x) = 0, \quad (8)$$

$$g(0, u(t, \cdot)) u(t, 0) = \int_0^\infty dx \beta(x, u(t, \cdot)) u(t, x). \quad (9)$$

In particular, Eq. (9) gives the newborns.

The *generalized net reproduction rate* is defined as

$$R(u) = \int_0^\infty \beta(x, u) \Pi(x, u) dx, \quad (10)$$

where $\Pi(x, u) = \frac{1}{g(x, u)} e^{-\int_0^x \frac{\mu(y, u)}{g(y, u)} dy}$ is an auxiliary function, said *generalized survival probability* and it represents a stationary solution of Eq. (8), i. e. the differential part of the model.

In general β and Π depend on u in a functional way: for instance in Calsina and Saldana [2, 3] the dependence is given through a weighted integral; in my paper [1] the dependence is infinite-dimensional in a more general way, to manage hierarchical models.

Some examples are populations where fertility or mortality are influenced only by the immediately superior size: for instance a population of trees in a forest, where the contended resource is the light, that is intercepted by immediately taller trees than trees of size x but not by the trees that are very taller than x . (For a case of tree population model, see [7]).

A stationary solution u of (8)–(9) exists if and only if u satisfies the functional equation

$$u = G(u) \Pi(u), \quad (11)$$

where $G(u(\cdot)) = \int_0^\infty \beta(x', u(\cdot)) u(x') dx'$.

Eq. (11) is related to the condition $R(u) = 1$ that is used to prove the existence of nontrivial stationary solution (that is, nonzero). Under suitable regularity conditions, we have that $R(0) > 1$ is a sufficient condition.

With additional conditions on monotonicity of β/g and μ/g , the reproduction rate $R(u)$ is monotone decreasing and we exclude existence of nontrivial solution if $R(0) < 1$. This is a recurrent condition in dynamics of populations.

4 Other Recurrences of the Functional in Literature

Conditions on H and G in Prop. 1 are analogous to conditions given in [6], Teorema 2.1, b) Teorema [6] Let $-\infty < a < b \leq \infty$ and let h and g be positive functions on (a, b) , where g is continuous on (a, b) .

Assume that h is increasing on (a, b) and g is decreasing on (a, b) where $g(b^-) = 0$. Then, for any $p \in (0, 1]$,

$$\int_a^b h(x) d[-g(x)] \leq \left(\int_a^b h^p(x) d[-g^p(x)] \right)^p \quad (1.2) \quad (12)$$

If $1 \leq p < \infty$, then the inequality (1.2) holds in the reversed direction.

In [9], the theorem above extends from t^p to concave and convex functions ϕ , when they are positive and differentiable.

At the present I have no ideas if this fact would have any meaning for $R(u)$ or eventually estimates of it in the spaces L^p , however I think that the similarities of conditions is not a coincidence.

Heinig and Maligranda's original paper [6] treats monotone functions and Hölder inequalities on Hardy spaces. A related field can be about Fredholm-Volterra equations.

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